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LETTER TO THE EDITOR

Simple treatment of correlated multiplicative and additive noises

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Abstract. Two simple rules are derived for obtaining the solution of a problem with correlated multiplicative and additive noise in terms of the results for systems with non-correlated noise. It is shown that the mean free passage time decreases as the correlation time increases, and its influence becomes important for not-too-small correlation times.

A variety of phenomena in physics, chemistry and biology are modelled by the stochastic differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) + g(x)\xi(t) + \eta(t) \tag{1}$$

which in particular describes the overdamped one-dimensional motion of a particle in an external field $U(x)$, where $f(x) = -\frac{\mathrm{d}U}{\mathrm{d}x}$, subject to multiplicative noise $\xi(t)$ and additive noise $\eta(t)$ [1].

Additive noise arises, for example, from the fast dynamics of other (additional to x) degrees of freedom, or from the non-zero temperature of a system (thermal noise). Alternatively, multiplicative noise is related to the stochastic nature of external fields or boundary conditions. Both types of noise have been intensively studied in different problems, for example chemical reactions, electrical circuits, liquid crystals, hydrodynamic systems, lasers, plasmas, nuclear reactions and biological systems (see [2] and references therein).

The simplest assumption is that both $\xi(t)$ and $\eta(t)$ describe Gaussian white noise with zero mean and correlations

$$\langle\eta(t)\eta(s)\rangle = 2D\delta(t-s) \qquad \langle\xi(t)\xi(s)\rangle = 2\alpha\delta(t-s). \tag{2}$$

By use of the Stratanovich interpretation of the stochastic differential equation (1) [3], the Fokker–Planck equation for the probability function $P(x,t)$ corresponding to (1) and (2) is given by [4–7]

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}[A(x)P] + \frac{\partial^2}{\partial x^2}[B(x)P] \tag{3}$$

where

$$A(x) = f(x) + \alpha g(x)\frac{\mathrm{d}g}{\mathrm{d}x} \qquad B(x) = D + \alpha g^2(x). \tag{4}$$

An important special case of (1) is the bistable potential $U(x) = -\frac{ax^2}{2} + \frac{bx^4}{4}$ with stochastically varying barrier curvature, i.e., where

$$f(x) = ax - bx^3 \qquad g(x) = x. \tag{5}$$

In the absence of additive noise, $D = 0$, equations (3)–(5) can be solved exactly [8], while in the presence of both noises, $D \neq 0$ and $\alpha \neq 0$, the analysis has been restricted to the stationary solution, $P_{st}(x) \sim B^{-1}(x)\exp(-\int^x \frac{A(y)}{B(y)}\,\mathrm{d}y)$ and its first moments [7], the correlation function [7], and the mean free passage time (MFPT) [2]. In contrast to additive noise, multiplicative noise shifts the stationary points of a deterministic system (noise-induced transition [9]). Instead of the condition $f(x) = 0$, these points are determined from the extrema of $P_{st}(x)$, i.e. from the condition

$$f(x) - \frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}x}[g^2(x)] = 0 \tag{6}$$

which, in the case of the bistable potential (5), reduces to

$$bx^3 + (\alpha - a)x = 0. \tag{7}$$

Another important characteristic of the dynamics is the MFPT, $T(x_2; x_1)$, to reach the point x_2 starting from the point x_1 which is defined by $P_{st}(x)$ [1],

$$T(x_2; x_1) = \int_{x_1}^{x_2} \frac{\mathrm{d}x}{B(x)P_{st}(x)} \int_{-\infty}^{x} P_{st}(y)\,\mathrm{d}y. \tag{8}$$

The analysis of (8) for different values of D, α and the energy barrier has been performed in [4–6]. It turns out that additional multiplicative noise results in a decrease of MFPT compared with the MFPT for additive noise alone, i.e. multiplicative noise ‘helps’ the tunnelling through the barrier.

Up to this point, it has been assumed that additive and multiplicative noise are not correlated, i.e. $\langle \xi(t)\eta(s)\rangle = 0$. There are, however, some situations where the latter condition is violated. This happens when both noises have the same origin, as in laser dynamics [10], or when strong external noise leads to an appreciable change in the internal structure of a system and hence in internal noise. The influence of the correlation between noises on the dynamics of a system has recently become the objective of much research [11–20], some results being quite cumbersome.

In this letter we derive two simple rules for the description of the dynamics of a system with correlated noise in terms of that with no such correlations. Let us first assume that additive and multiplicative noise are delta-correlated with the parameter λ measuring the strength of these correlations:

$$\langle \xi(t)\eta(s)\rangle = 2\lambda\sqrt{\alpha D}\,\delta(t - s). \tag{9}$$

Rule 1. To obtain the Fokker–Planck equation for a system with correlated noise described by equations (1), (2) and (9) one replaces $g(x)$ and D in the Fokker–Planck equations (3) and (4) according to the following rule:

$$g(x) \longrightarrow g(x) + \lambda\sqrt{\frac{D}{\alpha}} \qquad D \to D(1 - \lambda^2) \tag{10}$$

(the case $\lambda = 1$ has to be considered separately).

The proof of (10) is based on the observation [21] that equation (1) can be rewritten as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) + \left[g(x) + \lambda\sqrt{\frac{D}{\alpha}}\right]\xi(t) + \varsigma(t) \tag{11}$$

where

$$\varsigma(t) = \eta(t) - \lambda\sqrt{\frac{D}{\alpha}}\xi(t). \tag{12}$$

One can easily confirm that $\langle \varsigma(t)\varsigma(s)\rangle = 2D(1-\lambda^2)\delta(t-s)$ and $\langle \varsigma(t)\xi(s)\rangle = 0$, i.e., the problem with correlated noise is reduced to the original problem of non-correlated noises and becomes identical to it after performing the transformations (10).

Hence, in the case of delta-correlated noise, the functions $A(x)$ and $B(x)$ in the Fokker–Planck equation (3) can be obtained by inserting (10) into (4), which gives

$$A(x) = f(x) + \alpha\left[g(x) + \lambda\sqrt{\frac{D}{\alpha}}\right]\frac{\mathrm{d}g}{\mathrm{d}x} \qquad B(x) = D + \alpha g^2(x) + 2\lambda\sqrt{D\alpha}\,g(x) \tag{13}$$

or, for the bistable potential (5),

$$A(x) = (a+\alpha)x - bx^3 + \lambda\sqrt{D\alpha} \qquad B(x) = D + \alpha x^2 + 2\lambda\sqrt{D\alpha}\,x. \tag{14}$$

The extrema of $P_{st}(x)$ for the correlated noise can be found by inserting (10) into (6) or (7), which results in

$$f(x) - \frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left[g(x) + \lambda\sqrt{\frac{D}{\alpha}}\right]^2 = 0 \tag{15}$$

and

$$(a-\alpha)x - bx^3 - \lambda\sqrt{D\alpha} = 0. \tag{16}$$

In contrast to the case of non-correlated noise, the positions of the extrema now depend on the additive noise as well. The MFPT defined by equations (8) and (13) increases with λ.

Equations (13)–(16) are identical to the appropriate equations of [15] which have been obtained after a quite complicated derivation of the Fokker–Planck equation for correlated noise rather than by the simple substitution of (10).

So far we have assumed delta-correlations between multiplicative and additive noise. More complicated 'coloured' correlation is characterized by the non-zero correlations time τ. We consider the exponential Gaussian correlation of the form

$$\langle \eta(t)\xi(s)\rangle = \frac{\lambda\sqrt{D\alpha}}{\tau}\exp\left(-\frac{|t-s|}{\tau}\right) \tag{17}$$

while the autocorrelation functions (2) remain delta-correlated.

Derivation of the Fokker–Planck equation for coloured noise is a complicated problem [3, 22] which has an exact solution only in the simplest cases. For example, for the linear stochastic equation,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -ax + \zeta(t) + \psi(t) \tag{18}$$

where $\psi(t)$ is the white noise of strength D and $\zeta(t)$ is the exponentially correlated Gaussian noise

$$\langle \zeta(t)\zeta(s)\rangle = \frac{Q}{\tau}\exp\left(-\frac{|t-s|}{\tau}\right) \tag{19}$$

the exact stationary solution of the Fokker–Planck equation has been found [6]. It turns out that this solution can be found in terms of that for the case of two white noise with strengths D and Q by the simple transformation $Q \to \frac{Q}{1+a\tau}$. Quite surprisingly, such a simple rule also applies to our case. Indeed, if one compares the approximate Fokker–Planck equation (equation (22) in [19]) for the case (17) of the exponentially correlated multiple and additive noises with the Fokker–Planck equations (3), (13) for the case of delta-correlations, one sees that the former can be obtained from the latter by the simple replacement $\lambda \longrightarrow \frac{\lambda}{1+2a\tau}$. Therefore, at least in the framework of the Hanggi *et al* [23] approximate procedure used in [19], one arrives at the following.

Rule 2. In order to derive the Fokker–Planck equation for a system with exponentially correlated multiplicative and additive noise (equation (17)), one replaces λ in the Fokker–Planck equation for the delta-correlated noise (equation (13)) according to the following rule:

$$\lambda \to \frac{\lambda}{1 + 2a\tau}. \tag{20}$$

Renormalizations similar to (20) have been used before for comparison of coloured and white noise when either multiplicative or additive noise was present. Path integral analysis shows [24] that the stationary distribution for the linear problem ($f(x) = -ax$, $g(x) = 1$, $\eta = 0$ in (1)) with coloured multiplicative noise can be obtained from that with white noise by replacing α by $\alpha(1 + a\tau)^{-1}$. Likewise, in the adiabatic approximation for a small noise intensity of additive noise, ($g = 0$ in equation (1)), the influence of the correlation time τ can be taken into account [25] by replacing D by $D\,[1 - \tau\langle f'(x)\rangle]^{-1}$. Similar renormalizations can be performed for the underdamped motion [26].

Substituting (20) into (16) yields the extrema of $P_{st}(x)$ in the case being considered, which was previously obtained by quite a complicated way in [19]. However, the MFPT has not yet been calculated. The explicit expression for the MFPT for correlations of the form (17) can be obtained by using the transformation (20) in the MFPT for the delta-correlated case (equation (15) in [19])). Without rewriting this cumbersome formula, one immediately concludes from (20) that:

(1) The correlation time τ influences the MFPT in the opposite direction compared with λ, i.e. the MFPT decreases when τ increases.
(2) The influence of τ becomes important for not-too-small τ ($\gg \frac{1}{2a}$), i.e., when τ becomes large compared with the characteristic time a^{-1} of the deterministic motion.

In conclusion, it is shown that the influence of the correlations between multiplicative and additive noise on the stastistical properties of classical systems can be easily found from the known properties of non-correlated systems. The MFPT is considered as an example.

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